GLOBAL CHAOS SYNCHRONIZATION OF HYPERCHAOTIC PANG AND HYPERCHAOTIC WANG SYSTEMS VIA ADAPTIVE CONTROL

Sundarapandian Vaidyanathan[1] and Karthikeyan Rajagopal[2]

[1]Research and Development Centre, Vel Tech Dr. RR & Dr. SR Technical University
Avadi, Chennai-600 062, Tamil Nadu, INDIA
sundarvtu@gmail.com
[2]School of Electronics and Electrical Engineering, Singhania University
Dist. Jhunjhunu, Rajasthan-333 515, INDIA
rkarthiekeyan@gmail.com

ABSTRACT

This paper investigates the global chaos synchronization of identical hyperchaotic Wang systems, identical hyperchaotic Pang systems, and non-identical hyperchaotic Wang and hyperchaotic Pang systems via adaptive control method. Hyperchaotic Pang system (Pang and Liu, 2011) and hyperchaotic Wang system (Wang and Liu, 2006) are recently discovered hyperchaotic systems. Adaptive control method is deployed in this paper for the general case when the system parameters are unknown. Sufficient conditions for global chaos synchronization of identical hyperchaotic Pang systems, identical hyperchaotic Wang systems and non-identical hyperchaotic Pang and Wang systems are derived via adaptive control theory and Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is very convenient for the global chaos synchronization of the hyperchaotic systems discussed in this paper. Numerical simulations are presented to validate and demonstrate the effectiveness of the proposed synchronization schemes.

KEYWORDS

Adaptive Control, Hyperchaos, Synchronization, Hyperchaotic Pang System, Hyperchaotic Wang System.

1. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1]. Since chaos phenomenon in weather models was first observed by Lorenz in 1963 [2], a large number of chaos phenomena and chaos behaviour have been discovered in physical, social, economical, biological and electrical systems.

A hyperchaotic system is usually characterized as a chaotic system with more than one positive Lyapunov exponent implying that the dynamics expand in more than one direction giving rise to "thicker" and "more complex" chaotic dynamics. The first hyperchaotic system was discovered by Rössler in 1979 [3].

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Chaos is an interesting nonlinear phenomenon and has been extensively studied in the last two decades [1-40].

Synchronization of chaotic systems is a phenomenon which may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a very challenging problem.

In 1990, Pecora and Carroll [4] deployed control techniques to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has been widely explored in a variety of fields including physical systems [5], chemical systems [6], ecological systems [7], secure communications [8-10], etc.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the seminal work by Pecora and Carroll [4], a variety of impressive approaches have been proposed for the synchronization of chaotic systems such as the OGY method [11], active control method [12-16], adaptive control method [17-22], sampled-data feedback synchronization method [23], time-delay feedback method [24], backstepping method [25-26], sliding mode control method [27-32], etc.

In this paper, we investigate the global chaos synchronization of uncertain hyperchaotic systems, *viz.* identical hyperchaotic Pang systems ([33], 2011), identical hyperchaotic Wang systems ([34], 2006) and non-identical hyperchaotic Pang and hyperchaotic Wang systems. We consider the general case when the parameters of the hyperchaotic systems are unknown.

This paper is organized as follows. In Section 2, we provide a description of the hyperchaotic systems addressed in this paper, *viz.* hyperchaotic Pang system (2011) and hyperchaotic Wang system (2006). In Section 3, we discuss the adaptive synchronization of identical hyperchaotic Pang systems. In Section 4, we discuss the adaptive synchronization of identical hyperchaotic Wang systems. In Section 5, we discuss the adaptive synchronization of non-identical hyperchaotic Pang and hyperchaotic Wang systems. In Section 6, we summarize the main results obtained in this paper.

2. SYSTEMS DESCRIPTION

The hyperchaotic Pang system ([33], 2011) is described by the dynamics

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_2 - x_1 x_3 + x_4 \\
\dot{x}_3 &= -bx_3 + x_1 x_2 \\
\dot{x}_4 &= -d(x_1 + x_2)
\end{aligned} \tag{1}$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, c, d are positive, constant parameters of the system.

The 4-D system (1) is hyperchaotic when the parameter values are taken as

$$a = 36, \quad b = 3, \quad c = 20 \quad \text{and} \quad d = 2$$

The state orbits of the hyperchaotic Pang chaotic system (1) are shown in Figure 1.

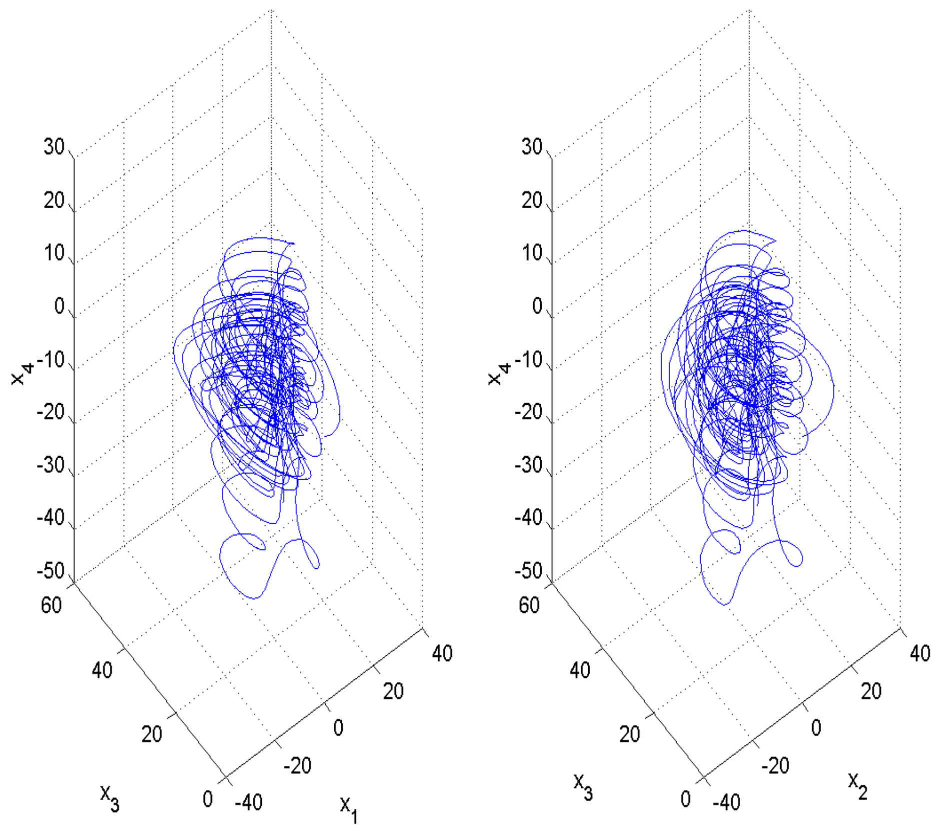

Figure 1. State Orbits of the Hyperchaotic Pang Chaotic System

The hyperchaotic Wang system ([34], 2006) is described by

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) \\
\dot{x}_2 &= \beta x_1 - x_1 x_3 + x_4 \\
\dot{x}_3 &= -\gamma x_3 + \varepsilon x_1^2 \\
\dot{x}_4 &= -\delta x_1
\end{aligned} \tag{2}$$

where x_1, x_2, x_3, x_4 are the state variables and $\alpha, \beta, \gamma, \delta, \varepsilon$ are positive constant parameters of the system.

The 4-D system (2) is hyperchaotic when the parameter values are taken as

$\alpha = 10, \;\; \beta = 40, \;\; \gamma = 2.5, \;\; \delta = 10.6 \;\; \text{and} \;\; \varepsilon = 4$

The state orbits of the hyperchaotic Wang chaotic system (2) are shown in Figure 2.

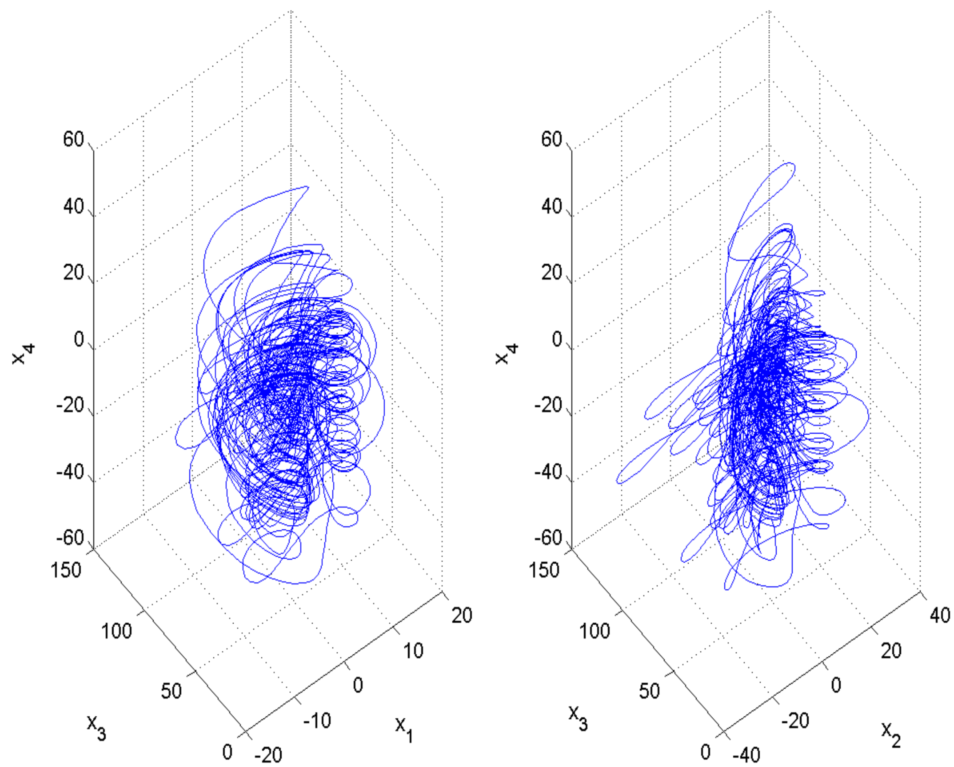

Figure 2. State Orbits of the Hyperchaotic Wang System

3. Adaptive Synchronization of Identical Hyperchaotic Pang Systems

3.1 Theoretical Results

In this section, we deploy adaptive control to achieve new results for the global chaos synchronization of identical hyperchaotic Pang systems ([33], 2011), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Pang dynamics described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= cx_2 - x_1 x_3 + x_4 \\ \dot{x}_3 &= -bx_3 + x_1 x_2 \\ \dot{x}_4 &= -d(x_1 + x_2) \end{aligned} \tag{3}$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, c, d are unknown, real ,constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Pang dynamics described by

$$\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + u_1 \\
\dot{y}_2 &= cy_2 - y_1 y_3 + y_4 + u_2 \\
\dot{y}_3 &= -by_3 + y_1 y_2 + u_3 \\
\dot{y}_4 &= -d(y_1 + y_2) + u_4
\end{aligned} \tag{4}$$

where y_1, y_2, y_3, y_4 are the state variables and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{5}$$

The error dynamics is easily obtained as

$$\begin{aligned}
\dot{e}_1 &= a(e_2 - e_1) + u_1 \\
\dot{e}_2 &= ce_2 + e_4 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= -be_3 + y_1 y_2 - x_1 x_2 + u_3 \\
\dot{e}_4 &= -d(e_1 + e_2) + u_4
\end{aligned} \tag{6}$$

Let us now define the adaptive control functions

$$\begin{aligned}
u_1(t) &= -\hat{a}(e_2 - e_1) - k_1 e_1 \\
u_2(t) &= -\hat{c}e_2 - e_4 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\
u_3(t) &= \hat{b}e_3 + y_1 y_2 - x_1 x_2 - k_3 e_3 \\
u_4(t) &= \hat{d}(e_1 + e_2) - k_4 e_4
\end{aligned} \tag{7}$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are estimates of a, b, c and d, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (7) into (6), the error dynamics simplifies to

$$\begin{aligned}
\dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= (c - \hat{c})e_2 - k_2 e_2 \\
\dot{e}_3 &= -(b - \hat{b})e_3 - k_3 e_3 \\
\dot{e}_4 &= -(d - \hat{d})(e_1 + e_2) - k_4 e_4
\end{aligned} \tag{8}$$

Let us now define the parameter estimation errors as

$$e_a = a - \hat{a}, \quad e_b = b - \hat{b}, \quad e_c = c - \hat{c} \quad \text{and} \quad e_d = d - \hat{d} \tag{9}$$

Substituting (9) into (8), we obtain the error dynamics as

$$\begin{aligned}
\dot{e}_1 &= e_a(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= e_c e_2 - k_2 e_2 \\
\dot{e}_3 &= -e_b e_3 - k_3 e_3 \\
\dot{e}_4 &= -e_d(e_1 + e_2) - k_4 e_4
\end{aligned} \tag{10}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_4, e_a, e_b, e_c, e_d) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right), \tag{11}$$

which is a positive definite function on R^8.

We also note that

$$\dot{e}_a = -\dot{\hat{a}}, \; \dot{e}_b = -\dot{\hat{b}}, \; \dot{e}_c = -\dot{\hat{c}} \quad \text{and} \quad \dot{e}_d = -\dot{\hat{d}} \tag{12}$$

Differentiating (11) along the trajectories of (10) and using (12), we obtain

$$\begin{aligned}
\dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a\left[e_1(e_2 - e_1) - \dot{\hat{a}}\right] + e_b\left[-e_3^2 - \dot{\hat{b}}\right] \\
&+ e_c\left[e_2^2 - \dot{\hat{c}}\right] + e_d\left[-e_4(e_1 + e_2) - \dot{\hat{d}}\right]
\end{aligned} \tag{13}$$

In view of Eq. (13), the estimated parameters are updated by the following law:

$$\begin{aligned}
\dot{\hat{a}} &= e_1(e_2 - e_1) + k_5 e_a \\
\dot{\hat{b}} &= -e_3^2 + k_6 e_b \\
\dot{\hat{c}} &= e_2^2 + k_7 e_c \\
\dot{\hat{d}} &= -e_4(e_1 + e_2) + k_8 e_d
\end{aligned} \tag{14}$$

where k_4, k_5, k_6 and k_7 are positive constants.

Substituting (14) into (13), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 \tag{15}$$

which is a negative definite function on R^8.

Thus, by Lyapunov stability theory [35], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error e_a, e_b, e_c, e_d decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 1. *The identical hyperchaotic Pang systems (3) and (4) with unknown parameters are globally and exponentially synchronized via the adaptive control law (7), where the update law for the parameter estimates is given by (14) and* $k_i, (i = 1, 2, \ldots, 8)$ *are positive constants. Also, the parameter estimates* $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ *and* $\hat{d}(t)$ *exponentially converge to the original values of the parameters* a, b, c *and* d, *respectively, as* $t \to \infty$. ■

3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (3) and (4) with the adaptive control law (14) and the parameter update law (14) using MATLAB.

We take

$$k_i = 4 \text{ for } i = 1, 2, \ldots, 8.$$

For the hyperchaotic Pang systems (3) and (4), the parameter values are taken as

$$a = 36, \quad b = 3, \quad c = 20, \quad d = 2$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 12, \quad \hat{b}(0) = 4, \quad \hat{c}(0) = 2, \quad \hat{d}(0) = 21$$

The initial values of the master system (3) are taken as

$$x_1(0) = 12, \quad x_2(0) = 18, \quad x_3(0) = 35, \quad x_4(0) = 6$$

The initial values of the slave system (4) are taken as

$$y_1(0) = 20, \quad y_2(0) = 5, \quad y_3(0) = 16, \quad y_4(0) = 22$$

Figure 3 depicts the global chaos synchronization of the identical hyperchaotic Pang systems (3) and (4).

Figure 4 shows that the estimated values of the parameters, viz. $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ and $\hat{d}(t)$ converge exponentially to the system parameters

$$a = 36, \quad b = 3, \quad c = 20 \quad \text{and} \quad d = 2$$

as $t \to \infty$.

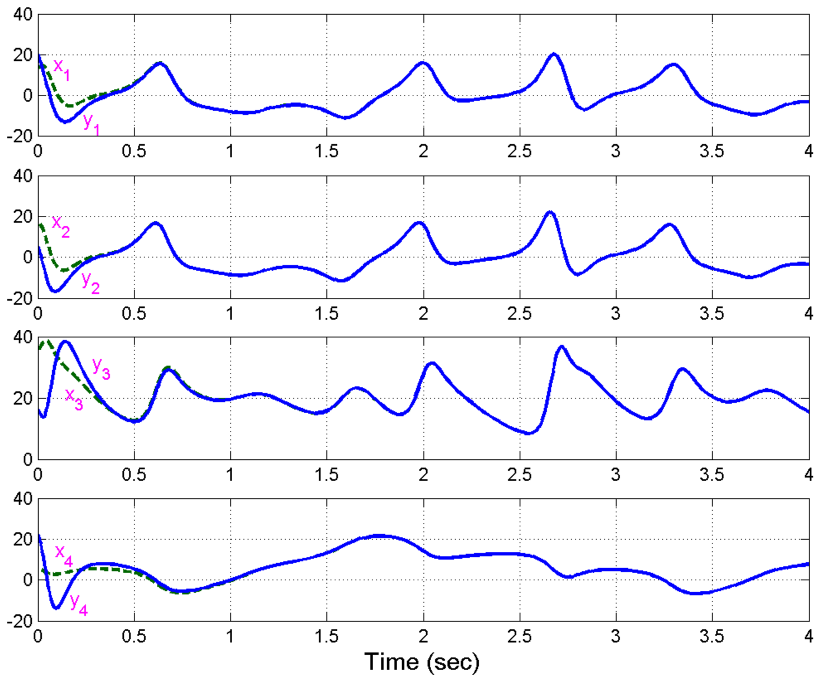

Figure 3. Complete Synchronization of Hyperchaotic Pang Systems

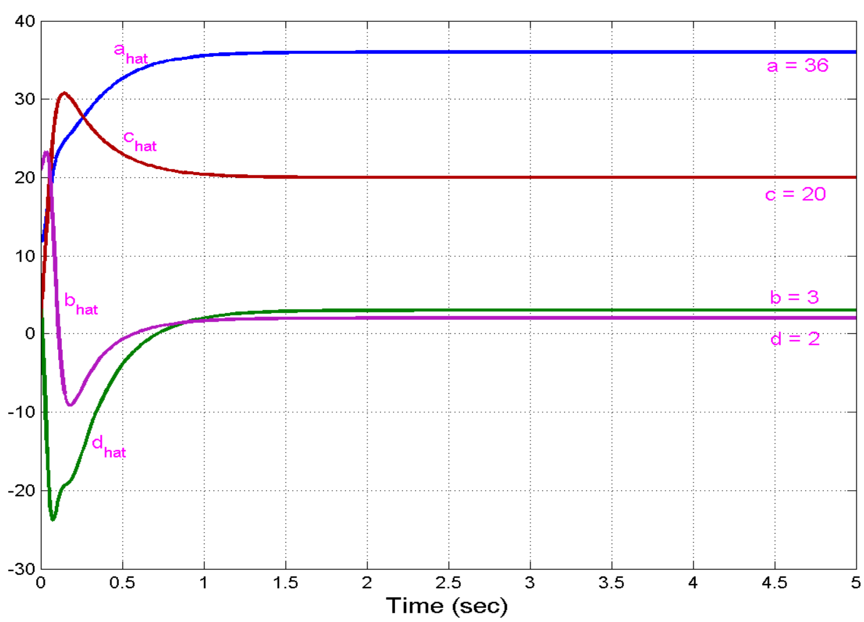

Figure 4. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$

4. ADAPTIVE SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC WANG SYSTEMS

4.1 Theoretical Results

In this section, we deploy adaptive control to achieve new results for the global chaos synchronization of identical hyperchaotic Wang systems ([34], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Wang dynamics described by

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) \\
\dot{x}_2 &= \beta x_1 - x_1 x_3 + x_4 \\
\dot{x}_3 &= -\gamma x_3 + \varepsilon x_1^2 \\
\dot{x}_4 &= -\delta x_1
\end{aligned} \tag{16}$$

where x_1, x_2, x_3, x_4 are the state variables and $\alpha, \beta, \gamma, \delta, \varepsilon$ are unknown, real ,constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Wang dynamics described by

$$\begin{aligned}
\dot{y}_1 &= \alpha(y_2 - y_1) + u_1 \\
\dot{y}_2 &= \beta y_1 - y_1 y_3 + y_4 + u_2 \\
\dot{y}_3 &= -\gamma y_3 + \varepsilon y_1^2 + u_3 \\
\dot{y}_4 &= -\delta y_1 + u_4
\end{aligned} \tag{17}$$

where y_1, y_2, y_3, y_4 are the state variables and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$\begin{aligned}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3 \\
e_4 &= y_4 - x_4
\end{aligned} \tag{18}$$

The error dynamics is easily obtained as

$$\begin{aligned}
\dot{e}_1 &= \alpha(e_2 - e_1) + u_1 \\
\dot{e}_2 &= \beta e_1 + e_4 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= -\gamma e_3 + \varepsilon(y_1^2 - x_1^2) + u_3 \\
\dot{e}_4 &= -\delta e_1 + u_4
\end{aligned} \tag{19}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{\alpha}(e_2 - e_1) - k_1 e_1 \\ u_2(t) &= -\hat{\beta} e_1 - e_4 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\gamma} e_3 - \hat{\varepsilon}(y_1^2 - x_1^2) - k_3 e_3 \\ u_4(t) &= \hat{\delta} e_1 - k_4 e_4 \end{aligned} \tag{20}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ and $\hat{\varepsilon}$ are estimates of $\alpha, \beta, \gamma, \delta$ and ε, respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (20) into (19), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (\beta - \hat{\beta}) e_1 - k_2 e_2 \\ \dot{e}_3 &= -(\gamma - \hat{\gamma}) e_3 + (\varepsilon - \hat{\varepsilon})(y_1^2 - x_1^2) - k_3 e_3 \\ \dot{e}_4 &= -(\delta - \hat{\delta}) e_1 - k_4 e_4 \end{aligned} \tag{21}$$

Let us now define the parameter estimation errors as

$$e_\alpha = \alpha - \hat{\alpha},\ e_\beta = \beta - \hat{\beta},\ e_\gamma = \gamma - \hat{\gamma},\ e_\delta = \delta - \hat{\delta} \text{ and } e_\varepsilon = \varepsilon - \hat{\varepsilon} \tag{22}$$

Substituting (22) into (21), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_\beta e_1 - k_2 e_2 \\ \dot{e}_3 &= -e_\gamma e_3 + e_\varepsilon (y_1^2 - x_1^2) - k_3 e_3 \\ \dot{e}_4 &= -e_\delta e_1 - k_4 e_4 \end{aligned} \tag{23}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V(e_1, e_2, e_3, e_4, e_\alpha, e_\beta, e_\gamma, e_\delta, e_\varepsilon) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_\varepsilon^2 \right), \tag{24}$$

which is a positive definite function on R^9.

We also note that

$$\dot{e}_\alpha = -\dot{\hat{\alpha}},\ \dot{e}_\beta = -\dot{\hat{\beta}},\ \dot{e}_\gamma = -\dot{\hat{\gamma}}, \dot{e}_\delta = -\dot{\hat{\delta}} \text{ and } \dot{e}_\varepsilon = -\dot{\hat{\varepsilon}} \tag{25}$$

Differentiating (24) along the trajectories of (23) and using (25), we obtain

$$\begin{aligned}\dot{V} &= -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_\alpha \left[e_1(e_2 - e_1) - \dot{\hat{\alpha}} \right] + e_\beta \left[e_1 e_2 - \dot{\hat{\beta}} \right] \\ &\quad + e_\gamma \left[-e_3^2 - \dot{\hat{\gamma}} \right] + e_\delta \left[-e_1 e_4 - \dot{\hat{\delta}} \right] + e_\varepsilon \left[e_3 (y_1^2 - x_1^2) - \dot{\hat{\varepsilon}} \right]\end{aligned} \tag{26}$$

In view of Eq. (26), the estimated parameters are updated by the following law:

$$\begin{aligned}\dot{\hat{\alpha}} &= e_1(e_2 - e_1) + k_5 e_\alpha \\ \dot{\hat{\beta}} &= e_1 e_2 + k_6 e_\beta \\ \dot{\hat{\gamma}} &= -e_3^2 + k_7 e_\gamma \\ \dot{\hat{\delta}} &= -e_1 e_4 + k_8 e_\delta \\ \dot{\hat{\varepsilon}} &= e_3 (y_1^2 - x_1^2) + k_9 e_\varepsilon\end{aligned} \tag{27}$$

where $k_i, \ (i = 5, \ldots, 9)$ are positive constants.

Substituting (27) into (26), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_\alpha^2 - k_6 e_\beta^2 - k_7 e_\gamma^2 - k_8 e_\delta^2 - k_9 e_\varepsilon^2 \tag{28}$$

which is a negative definite function on R^9.

Thus, by Lyapunov stability theory [35], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error $e_\alpha, e_\beta, e_\gamma, e_\delta, e_\varepsilon$ decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 2. *The identical hyperchaotic Wang systems (16) and (17) with unknown parameters are globally and exponentially synchronized via the adaptive control law (20), where the update law for the parameter estimates is given by (27) and* $k_i, (i = 1, 2, \ldots, 9)$ *are positive constants. Also, the parameter estimates* $\hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{\delta}(t)$ *and* $\hat{\varepsilon}(t)$ *exponentially converge to the original values of the parameters* $\alpha, \beta, \gamma, \delta$ *and* ε, *respectively, as* $t \to \infty$. ■

4.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (16) and (17) with the adaptive control law (20) and the parameter update law (27) using MATLAB.

We take

$$k_i = 4 \text{ for } i = 1, 2, \ldots, 9.$$

For the hyperchaotic Wang systems (16) and (17), the parameter values are taken as

$$\alpha = 10, \quad \beta = 40, \quad \gamma = 2.5, \quad \delta = 10.6, \quad \varepsilon = 4$$

Suppose that the initial values of the parameter estimates are

$$\hat{\alpha}(0) = 5, \quad \hat{\beta}(0) = 10, \quad \hat{\gamma}(0) = 7, \quad \hat{\delta}(0) = 14, \quad \hat{\varepsilon}(0) = 9$$

The initial values of the master system (16) are taken as

$$x_1(0) = 21, \quad x_2(0) = 7, \quad x_3(0) = 16, \quad x_4(0) = 18$$

The initial values of the slave system (17) are taken as

$$y_1(0) = 4, \quad y_2(0) = 25, \quad y_3(0) = 30, \quad y_4(0) = 11$$

Figure 5 depicts the global chaos synchronization of the identical hyperchaotic Wang systems (16) and (17).

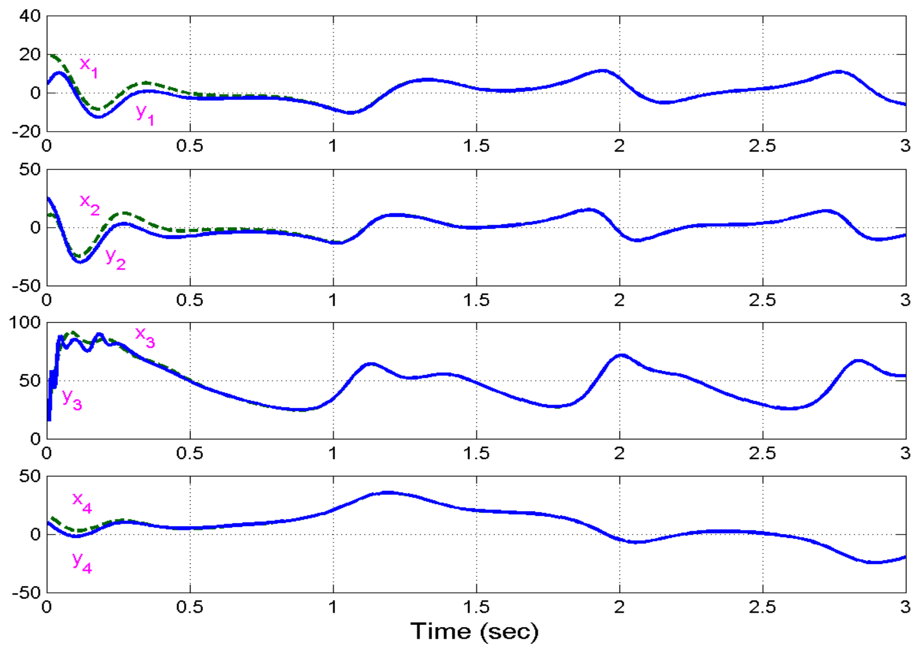

Figure 5. Complete Synchronization of Hyperchaotic Wang Systems

Figure 6 shows that the estimated values of the parameters, viz. $\hat{\alpha}(t)$, $\hat{\beta}(t)$, $\hat{\gamma}(t)$, $\hat{\delta}(t)$ and $\hat{\varepsilon}(t)$ converge exponentially to the system parameters $\alpha = 10$, $\beta = 40$, $\gamma = 2.5$, $\delta = 10.6$ and $\varepsilon = 4$ as $t \to \infty$.

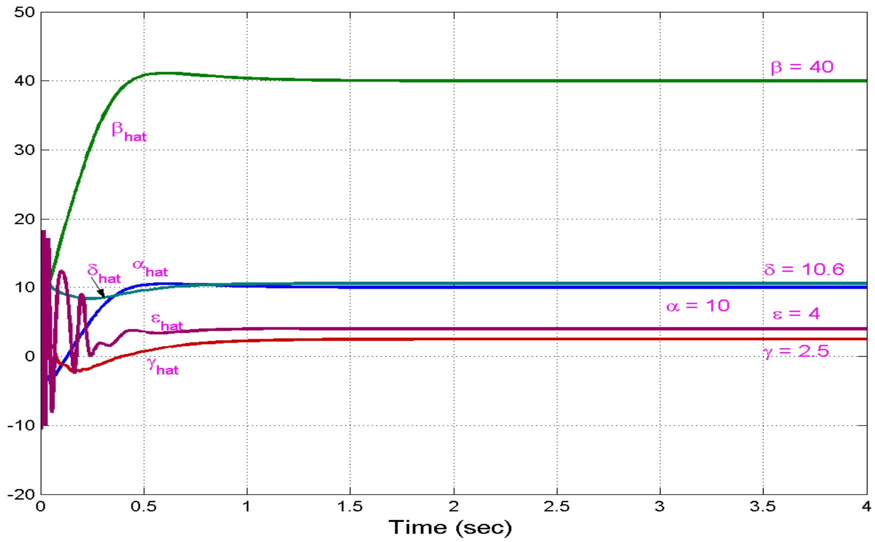

Figure 6. Parameter Estimates $\hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{\delta}(t), \hat{\varepsilon}(t)$

5. Adaptive Synchronization of Hyperchaotic Pang and Hyperchaotic Wang Systems

5.1 Theoretical Results

In this section, we discuss the global chaos synchronization of non-identical hyperchaotic Pang system ([33], 2011) and hyperchaotic Wang system ([34], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Pang system described by

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_2 - x_1x_3 + x_4 \\
\dot{x}_3 &= -bx_3 + x_1x_2 \\
\dot{x}_4 &= -d(x_1 + x_2)
\end{aligned} \tag{29}$$

where x_1, x_2, x_3, x_4 are the state variables and a, b, c, d are unknown, real ,constant parameters of the system.

As the slave system, we consider the controlled hyperchaotic Wang dynamics described by

$$\begin{aligned}
\dot{y}_1 &= \alpha(y_2 - y_1) + u_1 \\
\dot{y}_2 &= \beta y_1 - y_1y_3 + y_4 + u_2 \\
\dot{y}_3 &= -\gamma y_3 + \varepsilon y_1^2 + u_3 \\
\dot{y}_4 &= -\delta y_1 + u_4
\end{aligned} \tag{30}$$

where y_1, y_2, y_3, y_4 are the state variables , $\alpha,\beta,\gamma,\delta,\varepsilon$ are unknown, real, constant parameters of the system and u_1,u_2,u_3,u_4 are the nonlinear controllers to be designed.

The synchronization error is defined by

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \\ e_4 &= y_4 - x_4 \end{aligned} \tag{31}$$

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= \alpha(y_2 - y_1) - a(x_2 - x_1) + u_1 \\ \dot{e}_2 &= \beta y_1 - cx_2 + e_4 - y_1 y_3 + x_1 x_3 + u_2 \\ \dot{e}_3 &= -\gamma y_3 + bx_3 + \varepsilon y_1^2 - x_1 x_2 + u_3 \\ \dot{e}_4 &= -\delta y_1 + d(x_1 + x_2) + u_4 \end{aligned} \tag{32}$$

Let us now define the adaptive control functions

$$\begin{aligned} u_1(t) &= -\hat{\alpha}(y_2 - y_1) + \hat{a}(x_2 - x_1) - k_1 e_1 \\ u_2(t) &= -\hat{\beta} y_1 + \hat{c}x_2 - e_4 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\gamma} y_3 - \hat{b}x_3 - \hat{\varepsilon} y_1^2 + x_1 x_2 - k_3 e_3 \\ u_4(t) &= \hat{\delta} y_1 - \hat{d}(x_1 + x_2) - k_4 e_4 \end{aligned} \tag{33}$$

where $\hat{a},\hat{b},\hat{c},\hat{d}$, $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$ and $\hat{\varepsilon}$ are estimates of a,b,c,d, $\alpha,\beta,\gamma,\delta$ and ε, respectively, and $k_i, (i = 1,2,3,4)$ are positive constants.

Substituting (33) into (32), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\alpha - \hat{\alpha})(y_2 - y_1) - (a - \hat{a})(x_2 - x_1) - k_1 e_1 \\ \dot{e}_2 &= (\beta - \hat{\beta}) y_1 - (c - \hat{c}) x_2 - k_2 e_2 \\ \dot{e}_3 &= -(\gamma - \hat{\gamma}) y_3 + (b - \hat{b}) x_3 + (\varepsilon - \hat{\varepsilon}) y_1^2 - k_3 e_3 \\ \dot{e}_4 &= -(\delta - \hat{\delta}) y_1 + (d - \hat{d})(x_1 + x_2) - k_4 e_4 \end{aligned} \tag{34}$$

Let us now define the parameter estimation errors as

$$\begin{aligned} & e_a = a - \hat{a},\ \ e_b = b - \hat{b},\ \ e_c = c - \hat{c},\ \ e_d = d - \hat{d} \\ & e_\alpha = \alpha - \hat{\alpha},\ \ e_\beta = \beta - \hat{\beta},\ \ e_\gamma = \gamma - \hat{\gamma},\ e_\delta = \delta - \hat{\delta},\ e_\varepsilon = \varepsilon - \hat{\varepsilon} \end{aligned} \tag{35}$$

Substituting (35) into (34), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\alpha (y_2 - y_1) - e_a (x_2 - x_1) - k_1 e_1 \\ \dot{e}_2 &= e_\beta y_1 - e_c x_2 - k_2 e_2 \\ \dot{e}_3 &= -e_\gamma y_3 + e_b x_3 + e_\varepsilon y_1^2 - k_3 e_3 \\ \dot{e}_4 &= -e_\delta y_1 + e_d (x_1 + x_2) - k_4 e_4 \end{aligned} \tag{36}$$

We consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\delta^2 + e_\varepsilon^2\right), \tag{37}$$

which is a positive definite function on R^{13}.

We also note that

$$\begin{aligned} &\dot{e}_a = -\dot{\hat{a}}, \ \ \dot{e}_b = -\dot{\hat{b}}, \ \ \dot{e}_c = -\dot{\hat{c}}, \ \ \dot{e}_d = -\dot{\hat{d}} \\ &\dot{e}_\alpha = -\dot{\hat{\alpha}}, \ \dot{e}_\beta = -\dot{\hat{\beta}}, \ \dot{e}_\gamma = -\dot{\hat{\gamma}}, \ \dot{e}_\delta = -\dot{\hat{\delta}}, \ \dot{e}_\varepsilon = -\dot{\hat{\varepsilon}} \end{aligned} \tag{38}$$

Differentiating (37) along the trajectories of (36) and using (38), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_a\left[-e_1(x_2 - x_1) - \dot{\hat{a}}\right] + e_b\left[e_3 x_3 - \dot{\hat{b}}\right] + e_c\left[-e_2 x_2 - \dot{\hat{c}}\right] \\ &+ e_d\left[e_4(x_1 + x_2) - \dot{\hat{d}}\right] + e_\alpha\left[e_1(y_2 - y_1) - \dot{\hat{\alpha}}\right] + e_\beta\left[e_2 y_1 - \dot{\hat{\beta}}\right] + e_\gamma\left[-e_3 y_3 - \dot{\hat{\gamma}}\right] \\ &+ e_\delta\left[-e_4 y_1 - \dot{\hat{\delta}}\right] + e_\varepsilon\left[e_3 y_1^2 - \dot{\hat{\varepsilon}}\right] \end{aligned} \tag{39}$$

In view of Eq. (39), the estimated parameters are updated by the following law:

$$\begin{aligned} &\dot{\hat{a}} = -e_1(x_2 - x_1) + k_5 e_a, &&\dot{\hat{\alpha}} = e_1(y_2 - y_1) + k_9 e_\alpha \\ &\dot{\hat{b}} = e_3 x_3 + k_6 e_b, &&\dot{\hat{\beta}} = e_2 y_1 + k_{10} e_\beta \\ &\dot{\hat{c}} = -e_2 x_2 + k_7 e_c, &&\dot{\hat{\gamma}} = -e_3 y_3 + k_{11} e_\gamma \\ &\dot{\hat{d}} = e_4(x_1 + x_2) + k_8 e_d, &&\dot{\hat{\delta}} = -e_4 y_1 + k_{12} e_\delta \\ & &&\dot{\hat{\varepsilon}} = e_3 y_1^2 + k_{13} e_\varepsilon \end{aligned} \tag{40}$$

where $k_i, (i = 5, \ldots, 13)$ are positive constants.

Substituting (40) into (39), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_a^2 - k_6 e_b^2 - k_7 e_c^2 - k_8 e_d^2 - k_9 e_\alpha^2 - k_{10} e_\beta^2 \\ &- k_{11} e_\gamma^2 - k_{12} e_\delta^2 - k_{13} e_\varepsilon^2 \end{aligned} \tag{41}$$

which is a negative definite function on R^{13}.

Thus, by Lyapunov stability theory [35], it is immediate that the hybrid synchronization error $e_i, (i = 1, 2, 3, 4)$ and all the parameter estimation errors decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 3. *The non-identical hyperchaotic Pang system (29) and hyperchaotic Wang system (30) with unknown parameters are globally and exponentially synchronized via the adaptive control law (33), where the update law for the parameter estimates is given by (40) and* $k_i, (i = 1, 2, \ldots, 13)$ *are positive constants. Also, the parameter estimates* $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t), \hat{\delta}(t)$ *and* $\hat{\varepsilon}(t)$ *exponentially converge to the original values of the parameters* $a, b, c, d, \alpha, \beta, \gamma, \delta$ *and* ε, *respectively, as* $t \to \infty$. ■

5.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (29) and (30) with the adaptive control law (33) and the parameter update law (40) using MATLAB. We take $k_i = 4$ for $i = 1, 2, \ldots, 13$. For the hyperchaotic Pang and hyperchaotic Wang systems, the parameters of the systems are chosen so that the systems are hyperchaotic (see Section 2).

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 2,\ \hat{b}(0) = 5,\ \hat{c}(0) = 10,\ \hat{d}(0) = 12$$
$$\hat{\alpha}(0) = 7,\ \hat{\beta}(0) = 9,\ \hat{\gamma}(0) = 15,\ \hat{\delta}(0) = 22,\ \hat{\varepsilon}(0) = 25$$

The initial values of the master system (29) are taken as

$$x_1(0) = 27,\ \ x_2(0) = 11,\ \ x_3(0) = 28,\ \ x_4(0) = 6$$

The initial values of the slave system (30) are taken as

$$y_1(0) = 10,\ \ y_2(0) = 26,\ \ y_3(0) = 9,\ \ y_4(0) = 30$$

Figure 7 depicts the global chaos synchronization of hyperchaotic Pang and hyperchaotic Wang systems. Figure 8 shows that the estimated values of the parameters, viz. $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$, $\hat{d}(t)$, $\hat{\alpha}(t)$, $\hat{\beta}(t)$, $\hat{\gamma}(t)$, $\hat{\delta}(t)$ and $\hat{\varepsilon}(t)$ converge exponentially to the system parameters $a = 36$, $b = 3$, $c = 20$, $d = 2$, $\alpha = 10$, $\beta = 40$, $\gamma = 2.5$, $\delta = 10.6$ and $\varepsilon = 4$, respectively, as $t \to \infty$.

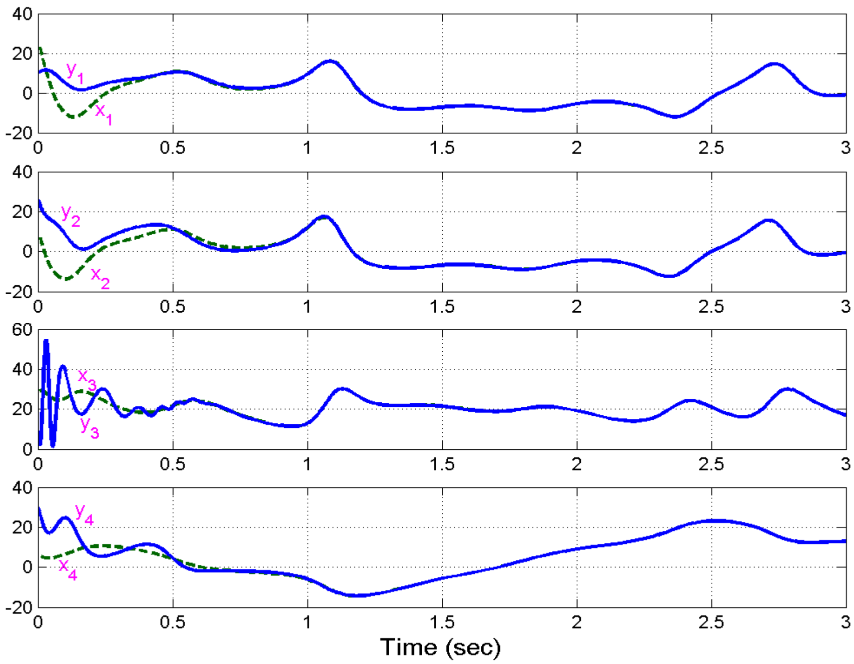

Figure 7. Complete Synchronization of Hyperchaotic Pang and Wang Systems

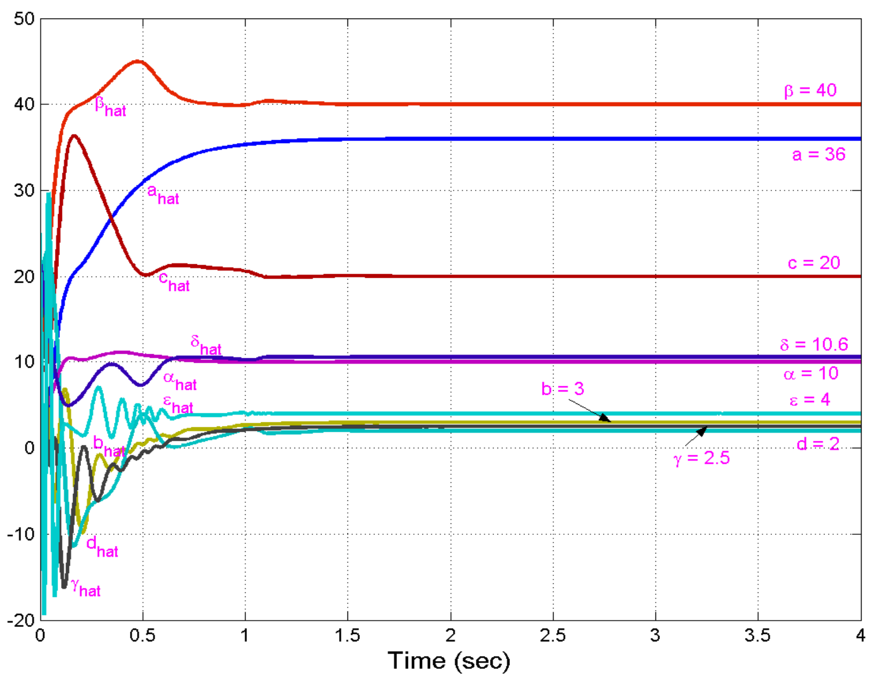

Figure 8. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t)$

6. CONCLUSIONS

In this paper, we have derived new results for the adaptive synchronization of identical hyperchaotic Pang systems (2011), identical hyperchaotic Wang systems (2006) and non-identical hyperchaotic Pang and hyperchaotic Wang systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is a very effective and convenient for achieving global chaos synchronization for the uncertain hyperchaotic systems discussed in this paper. Numerical simulations are given to illustrate the effectiveness of the adaptive synchronization schemes derived in this paper for the global chaos synchronization of identical and non-identical uncertain hyperchaotic Pang and hyperchaotic Wang systems.

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Authors

Dr. V. Sundarapandian obtained his Doctor of Science degree in Electrical and Systems Engineering from Washington University, Saint Louis, USA under the guidance of Late Dr. Christopher I. Byrnes (Dean, School of Engineering and Applied Science) in 1996. He is currently Professor in the Research and Development Centre at Vel Tech Dr. RR & Dr. SR Technical University, Chennai, Tamil Nadu, India. He has published over 190 refereed international publications. He has published over 100 papers in National Conferences and over 50 papers in International Conferences. He is the Editor-in-Chief of *International Journal of Mathematics and Scientific Computing*, *International Journal of Instrumentation and Control Systems, International Journal of Control Systems and Computer Modelling, International Journal of Information Technology, Control and Automation,* etc. His research interests are Linear and Nonlinear Control Systems, Chaos Theory and Control, Soft Computing, Optimal Control, Process Control, Operations Research, Mathematical Modelling, Scientific Computing using MATLAB etc. He has delivered several Key Note Lectures on Linear and Nonlinear Control Systems, Chaos Theory and Control, Scientific Computing using MATLAB/SCILAB, etc.

Mr. R. Karthikeyan obtained his M.Tech degree in Embedded Systems Technologies from Vinayaka Missions University, Tamil Nadu, India in 2007. He earned his B.E. degree in Electronics and Communication Engineering from Univeristy of Madras, Tamil Nadu, in 2005. He has published over 10 papers in refereed International Journals. He has published several papers on Embedded Systems in National and International Conferences. His current research interests are Em bedded Systems, Robotics, Communications and Control Systems.